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Symmetries of the stable Kneser graphs

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ABSTRACT

It is well known that the automorphism group of the Kneser graph $KG_{n,k}$ is the symmetric group on n letters. For $n \geqslant 2k+1$, $k \geqslant 2$, we prove that the automorphism group of the stable Kneser graph $SG_{n,k}$ is the dihedral group of order $2n$.

Let $[n] := [1, 2, 3, \ldots, n]$. For each $n \geqslant 2k$, $n, k \in \{1, 2, 3, \ldots\}$, the *Kneser graph* $KG_{n,k}$ has as vertices the k-subsets of $[n]$ with edges defined by disjoint pairs of k-subsets. For the same parameters, the *stable Kneser graph* $SG_{n,k}$ is the subgraph of $KG_{n,k}$ induced by the *stable* k-subsets of $[n]$, i.e. those subsets that do not contain any 2-subset of the form $\{i, i+1\}$ or $\{1, n\}$. The Kneser and stable Kneser graphs are important graphs in algebraic and topological combinatorics. L. Lovász proved in [3] via an ingenious use of the Borsuk–Ulam theorem that $\chi(KG_{n,k}) = n - 2k + 2$, verifying a conjecture due to M. Kneser. Shortly afterwards, A. Schrijver proved in [5], again using the Borsuk–Ulam theorem, that $\chi(SG_{n,k}) = n - 2k + 2$. Schrijver also proved that the stable Kneser graphs are vertex critical, i.e. the chromatic number of any proper subgraph of $SG_{n,k}$ is strictly less than $n - 2k + 2$; for this reason, the stable Kneser graphs are also known as the Schrijver graphs. These results sparked a series of dramatic applications of algebraic topology in combinatorics that continues to this day.

Despite these general advances, there are many unanswered questions regarding Kneser and stable Kneser graphs. For example, it is well known that for $n \geqslant 2k+1$ the automorphism group of the Kneser graph $KG_{n,k}$ is the symmetric group on n letters, with the action induced by the permutation action on $[n]$; see [2] for a textbook account. The proof of this relies on the Erdős–Ko–Rado theorem

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characterizing maximal independent sets in $KG_{n,k}$, where an *independent set* in a graph is a collection of pairwise non-adjacent vertices. However, the automorphism groups of the stable Kneser graphs are not determined; an exercise in [4] asks what can be said regarding the symmetries of these graphs. The purpose of this note is to prove the following.

Theorem 1. *For $n \geqslant 2k+1$ and $k \geqslant 2$, the automorphism group of $SG_{n,k}$ is isomorphic to the dihedral group of order $2n$.*

Our proof will have two cases: for $n = 2k+2$, we will find a distinguished complete bipartite subgraph of $SG_{2k+2,k}$ and analyze the action of $\mathrm{Aut}(SG_{2k+2,k})$ on it to obtain our result. For $n \neq 2k+2$, we will proceed in the same way as the standard proof for $KG_{n,k}$, with the role of the Erdős–Ko–Rado theorem played by the following. Let $\binom{[n]}{k}_{stab}$ denote the stable k-subsets of $[n]$.

Theorem 2. *(See J. Talbot [7].) For $n \geqslant 2k+1$, $n \neq 2k+2$, the maximal independent sets in $SG_{n,k}$ are all of the form $\{A \in \binom{[n]}{k}_{stab}\colon\ i \in A\}$ for a fixed $i \in [n]$.*

Theorem 2 is a very useful tool for investigating $SG_{n,k}$. For example, it is noted in the introduction to [6] that the fractional chromatic number of $SG_{n,k}$ is easily obtained using this result. The case $n = 2k+2$ is not covered by Theorem 2, hence our need for a separate proof in this case.

Proof of Theorem 1. Let D_{2n} denote the dihedral group of order $2n$. In our analysis, we will need to recognize D_{2n} as a subgroup of S_n, the symmetric group on n letters, in particular as the subgroup generated by the cycle $(12\cdots n)$ and the involution $i \mapsto n-i+2 \bmod n$. It is clear that D_{2n} injects into $\mathrm{Aut}(SG_{n,k})$, as D_{2n} acts on $SG_{n,k}$ by acting on $[n]$. Thus, $\mathrm{Aut}(SG_{n,k})$ contains at least a dihedral subgroup. We will now show that this is the full automorphism group.

We first handle two cases that do not fall within our analysis below. For the graphs $SG_{6,2}$ and $SG_{8,3}$, it can be checked using Brendan McKay's program *nauty*, [1], that the automorphism groups are of order 12 and 16, respectively, and hence are dihedral.

When $n \neq 2k+2$, given $i \in [n]$, let $\mathcal{A}_i := \{A \in \binom{[n]}{k}_{stab}\colon\ i \in A\}$. Theorem 2 ensures that these are the maximal independent sets in $SG_{n,k}$. Any automorphism of $SG_{n,k}$ must permute these maximal independent sets, hence there exists a homomorphism Φ from $\mathrm{Aut}(SG_{n,k})$ to S_n. Given a non-trivial element $\rho \in \mathrm{Aut}(SG_{n,k})$, there exists $v \in \binom{[n]}{k}_{stab}$ such that $\rho(v) \neq v$, i.e. there exists $j \in v$ such that $j \notin \rho(v)$. It follows that $v \in \mathcal{A}_j$, but $v \notin \mathcal{A}_{\Phi(\rho)(j)}$, hence $\Phi(\rho)$ is non-trivial. We may conclude that Φ is injective. Note that the image of $D_{2n} \subseteq \mathrm{Aut}(SG_{n,k})$ under Φ is generated as described above.

We claim that if $\sigma \notin \Phi(D_{2n})$, then there exist $i, j \in [n]$ such that $|i-j| \neq 1$ and $|\sigma(i) - \sigma(j)| = 1$. This is easily seen by considering $\sigma(l)$ for an arbitrary $l \in [n]$. Either one of $\sigma(l) \pm 1$ is the image of some m with $|m-l| \neq 1$, in which case we are done, or σ injects the pair $l-1$ and $l+1$ to the pair $\sigma(l)-1$ and $\sigma(l)+1$. In the latter case, we are done unless $|\sigma(l-2) - \sigma(l-1)| = 1$ and $|\sigma(l+2) - \sigma(l+1)| = 1$. We may proceed with this line of reasoning until we either find i and j as claimed or we find that σ is an element of $\Phi(D_{2n})$, contradicting our assumption.

Suppose now that $\Phi(\beta) \in \Phi(\mathrm{Aut}(SG_{n,k})) \setminus D_{2n}$, and i, j are as in the claim. Since $|i-j| \neq 1$ and $n \geqslant 2k+1$, it is easy to check that $\mathcal{A}_i \cap \mathcal{A}_j \neq \emptyset$. However, $|\Phi(\beta)(i) - \Phi(\beta)(j)| = 1$ implies that $\mathcal{A}_{\Phi(\beta)(i)} \cap \mathcal{A}_{\Phi(\beta)(j)} = \emptyset$. Thus, there exists $v \in \mathcal{A}_i \cap \mathcal{A}_j$ such that $\beta(v) \in \mathcal{A}_{\Phi(\beta)(i)} \cap \mathcal{A}_{\Phi(\beta)(j)} = \emptyset$, a contradiction, and we are done.

Next consider the case $n = 2k+2, k > 3$. Let G_1 be the set of all k-subsets of $\{1, 3, \ldots, 2k+1\}$ and G_2 be the set of all k-subsets of $\{2, 4, \ldots, 2k+2\}$. We claim that G_1 and G_2 contain the vertices of a unique complete bipartite subgraph $K_{k+1,k+1}$ contained in $SG_{2k+2,k}$. We now prove our claim: Let F_1 and F_2 denote the partition of the vertex set for any subgraph of $SG_{2k+2,k}$ isomorphic to $K_{k+1,k+1}$. Since our ground set has size $2k+2$ and each element of F_i has $k+1$ neighbors, the union of the stable sets contained in each F_i must be of size $k+1$. Further, F_1 and F_2 must be disjoint. Thus, without loss of generality, F_1 contains the stable k-subsets of $\{1, 3, 5, \ldots, 2k+1\}$ while F_2 contains the stable k-subsets of $\{2, 4, 6, \ldots, 2k+2\}$. As this is the only such possible partition, $\mathrm{Aut}(SG_{2k+2,k})$ must preserve this unique $K_{k+1,k+1}$.

Given $X \in \binom{[2k+2]}{k}_{stab}$, we may pair each element $i \in X$ with $i+1 \notin X$, under cyclic addition. By so doing, we exhaust $2k$ of the elements of $[2k+2]$. The remaining two elements in $[2k+2]$ are either consecutive, yielding a unique $i \in X$ such that $i+4 \in X$, or they are non-consecutive, yielding unique $i, j \in X$ such that $i < j$ and $i+3, j+3 \in X$. We may therefore uniquely identify each X of the first type by the label $X_{\{i,i+1\}}$ and each X of the second type as $X_{\{i,j\}}$ for the indices $i < j$ determined above. Note that $j - i$ is odd, as there is one gap of length 3 between i and $i+3$ in the elements of $X_{\{i,j\}}$ and all gaps between $i+3$ and j have length 2, which implies that $j = i+3+2r$ for some r.

Let $o\{k+1\}$ be equal to $k+1$ if $k+1$ is odd and k if $k+1$ is even. The set $V_1 := F_1 \cup F_2$ contains all the $X_{\{i,i+1\}}$; the remaining $X_{\{i,j\}}$ may be partitioned into classes $V_3, V_5, \ldots, V_{o\{k+1\}}$ such that V_l contains all $X_{\{i,j\}}$ such that $|i-j| = l$, where the absolute value denotes the minimum distance from i to j cyclically.

We claim that for each $3 \leqslant l \leqslant o\{k+1\} - 2$, each vertex $X_{\{i,j\}}$ in V_l has exactly four neighbors in $SG_{2k+2,k}$: $X_{\{i-1,j+1\}}$, $X_{\{i+1,j-1\}}$, $X_{\{i-1,j-1\}}$ and $X_{\{i+1,j+1\}}$. This is justified by noting that any neighbor $X_{\{m,n\}}$ of $X_{\{i,j\}}$ must be a disjoint stable k-subset and in order for the conditions $m, n \notin X_{\{i,j\}}$ and $m+3, n+3 \notin X_{\{i,j\}}$ to simultaneously hold, it must be that $|m-i| = 1 = |n-j|$. Further, $X_{\{i-1,j+1\}}$ and $X_{\{i+1,j-1\}}$ are each contained in one of V_{l+2} and V_{l-2}, while $X_{\{i-1,j-1\}}$ and $X_{\{i+1,j+1\}}$ are contained in V_l. Thus, for each $3 \leqslant l \leqslant o\{k+1\} - 2$, V_l induces a $(2k+2)$-cycle as a subgraph of $SG_{2k+2,k}$. Considering the case where $k+1$ even, hence $k = o\{k+1\}$, V_k induces a $(2k+2)$-cycle where each vertex $X_{\{i,j\}}$ has four neighbors, $X_{\{i-1,j-1\}}, X_{\{i+1,j+1\}}, X_{\{i-1,j+1\}}$ and $X_{\{i+1,j-1\}}$, three of them in V_k and one in V_{k-2}. If $k+1$ is odd, then V_{k+1} induces a $(k+1)$-cycle where each vertex $X_{\{i,j\}}$ has four neighbors, $X_{\{i-1,j-1\}}, X_{\{i+1,j+1\}} \in V_{k+1}$ and $X_{\{i-1,j+1\}}, X_{\{i+1,j-1\}} \in V_{k-1}$.

Finally, each vertex in V_1 has a unique neighbor in V_3 formed by the edge $\{X_{\{i,i+1\}}, X_{\{i-1,i+2\}}\}$, thus any automorphism of $SG_{2k+2,k}$ preserving the $K_{k+1,k+1}$ induced by V_1 must also preserve V_3. Hence, there is a homomorphism from $\mathrm{Aut}(SG_{2k+2,k})$ onto the automorphism group of the $(2k+2)$-cycle induced by V_3, i.e. the dihedral group of order $2(2k+2)$. We will complete our proof by showing that this homomorphism is injective.

Suppose $\rho \in \mathrm{Aut}(SG_{2k+2,k})$ fixes V_3; we will show that it fixes every vertex of $SG_{2k+2,k}$. Any automorphism that fixes V_3 pointwise must also fix all the elements of V_1 since each element of V_1 has a unique neighbor in V_3. Having assumed that ρ fixes V_3 and having observed that for $3 \leqslant l \leqslant o\{k+1\} - 2$ each vertex in V_l has unique neighbors in V_{l-2} and V_{l+2}, we may inductively proceed and derive that ρ must fix all the vertices in V_l for $1 \leqslant l \leqslant o\{k+1\} - 2$. The only vertices remaining to check are those in $V_{o\{k+1\}}$, which we handle in two cases.

For $k+1$ even, we saw earlier that each vertex $X_{\{i,i+k\}} \in V_k$ has exactly one neighbor $X_{\{i+1,i+k-1\}} \in V_{k-2}$. Thus, V_k must also be fixed pointwise by ρ. In the case $k+1$ odd, i.e. $o\{k+1\} = k+1$, V_{k+1} induces a $(k+1)$-cycle, each of whose vertices are connected to a unique pair of vertices in V_{k-1}. As the vertices in V_{k-1} are fixed, we see V_{k+1} must also be fixed. Hence, ρ is trivial and our homomorphism is injective as desired. □

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